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Linear Algebra and its Applications 429 (2008) 1719–1729

LINEAR ALGEBRA AND ITS APPLICATIONS

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Level characteristics corresponding to peripheral eigenvalues of a nonnegative matrix

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Received 7 September 2006; accepted 18 May 2008
Available online 3 July 2008
Submitted by H. Schneider

Abstract

In this paper, we give necessary and sufficient conditions for a set of Jordan blocks to correspond to the peripheral spectrum of a nonnegative matrix. For each eigenvalue, λ, the λ-level characteristic (with respect to the spectral radius) is defined. The necessary and sufficient conditions include a requirement that the λ-level characteristic is majorized by the λ-height characteristic. An algorithm which has been implemented in MATLAB is given to determine when a multiset of Jordan blocks corresponds to the peripheral spectrum of a nonnegative matrix. The algorithm is based on the necessary and sufficient conditions given in this paper.

Keywords: Nonnegative matrix; Eventually nonnegative matrix; Peripheral spectrum; Jordan form; Level characteristic; Height characteristic

1. Introduction

The Perron–Frobenius theorem is an important result which has prompted much research pertaining to spectral properties of nonnegative matrices. Many combinatorial properties of the spectrum of a nonnegative matrix, and its generalized eigenspace, are known (for example, see [1,2,8]). Many authors have contributed to this knowledge base, and there are many interesting papers on this topic.

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doi:10.1016/j.laa.2008.05.014

While introducing the idea of level sets associated with a nonnegative matrix, Richman and Schneider give results pertaining to the singular graph and Weyr characteristic of an M-matrix in [6] as does Rothblum in [7]. Hershkowitz and Schneider give necessary and sufficient conditions on the relation between the height and level characteristics corresponding to the spectral radius of a nonnegative matrix in [3]. For an overview of these results and many more on this topic, see the survey papers [2,8]. In particular, it has been shown that, if ν and η are two sequences of nonnegative integers, then there exists a nonnegative matrix A with height characteristic η, corresponding to the spectral radius, and level characteristic ν, corresponding to the spectral radius (with entries rewritten in decreasing order), if and only if ν is majorized by η.

Looking at extensions of Perron–Frobenius theory from a cone theoretic perspective, Tam and Schneider obtain a necessary and sufficient condition on the peripheral spectrum of a matrix for which there is a proper cone that the matrix leaves invariant, with the core of the matrix with respect to this cone being simplicial [10]. Tam noted that the condition is also necessary for a nonnegative matrix and called for a matrix theoretic proof of this necessity, with the question being formally posed in [9]. In [5], McDonald extends the necessary condition and offers a necessary and sufficient condition (the extended Tam–Schneider condition) for a multiset of Jordan blocks to correspond to the peripheral spectrum of a nonnegative matrix A. In this paper, we offer necessary and sufficient conditions which are equivalent to the extended Tam–Schneider condition, but which are more concise and point out the relation between level sets and the height characteristic of A corresponding to the spectral radius. For each eigenvalue λ in the peripheral spectrum of a nonnegative matrix A, we define the λ-level characteristic (with respect to the spectral radius) and show that the λ-level characteristic is majorized by the λ-height characteristic. This property and the requirement that the peripheral spectrum associated with each level must be a union of complete sets of roots of unity provide necessary and sufficient conditions on the peripheral spectrum of a nonnegative matrix. This result is formally stated as Theorem 3.8. In Section 4, an algorithm is presented which determines whether or not there exists a nonnegative matrix with peripheral spectrum corresponding to a given multiset $\mathscr{J}$ of Jordan blocks. The algorithm is based on the conditions given in Theorem 3.8 and has been implemented in MATLAB.

2. Standard definitions and notation

We begin with some standard definitions.

Let $A \in \mathbf{C}^{nn}$.

For any $c \in \mathbf{C}$, we write $\bar{c}$ to represent the complex conjugate of c. For a matrix A, we write $\overline{A}$ to represent the matrix formed from A by conjugating each entry.

We will write $\langle n \rangle$ for $\{1, \ldots, n\}$.

We write $\mathbf{Z}_q = \{1, \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}}, \mathrm{e}^{\frac{4\pi \mathrm{i}}{q}}, \ldots, \mathrm{e}^{\frac{2(q-1)\pi \mathrm{i}}{q}}\}$ and refer to such a set as a *complete set of roots of unity*.

We let the multiset

$$\sigma(A) = \{\lambda | \lambda \text{ is an eigenvalue of } A\},$$

where each eigenvalue is listed the number of times it occurs as a root of the characteristic polynomial, and refer to it as the *spectrum* of A. We call

$$\rho(A) = \max_{\lambda \in \sigma(A)} \{|\lambda|\}$$

the *spectral radius* of A. The multiset

$$\pi(A) = \{\lambda \in \sigma(A) \mid |\lambda| = \rho(A)\}$$

is referred to as the *peripheral spectrum* of A. We let $\text{mult}_\lambda(A)$ denote the degree of λ as a root of the characteristic polynomial, and $\text{index}_\lambda(A)$ denote the degree of λ as a root of the minimal polynomial. Let $m = \text{index}_0(A)$, and for each $i \in \langle m\rangle$, set $\eta_i(A) = \text{nullity}(A^i) - \text{nullity}(A^{i-1})$. The sequence $\eta(A) = (\eta_1(A), \eta_2(A), \ldots, \eta_m(A))$ is referred to as the *height* or *Weyr characteristic* of A. We refer to the height characteristic of $A - \lambda I$ as the λ*-height characteristic of* A and denote it by $\eta_\lambda(A)$. If $\mathscr{J}$ is a collection of matrices, we let J be the direct sum of the matrices from $\mathscr{J}$ and define the λ-height characteristic of $\mathscr{J}$ to be the λ-height characteristic of J.

Let $\eta = (\eta_1, \eta_2, \ldots, \eta_t)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_t)$ be two sequences of nonnegative integers (append zeros if necessary to the end of the shorter sequence so that they are of the same length). We say that ν is *majorized* by η if $\sum_{i=1}^{j} \nu_i \leqslant \sum_{i=1}^{j} \eta_i$, for all $1 \leqslant j \leqslant t$, and $\sum_{i=1}^{t} \nu_i = \sum_{i=1}^{t} \eta_i$. We write $\nu \preceq \eta$. We denote by $\hat{\nu}$ the sequence ν reordered in decreasing order.

We write $J_j(\lambda)$ to represent the $j \times j$ matrix whose diagonal elements are λ, whose first subdiagonal elements are 1, and all other elements are zero. We will refer to such a matrix as a *Jordan block* (*with eigenvalue* λ).

We say a collection of Jordan blocks $\mathscr{J}$ is self-conjugate if, whenever $J_j(\lambda) \in \mathscr{J}$ and λ is complex, then $J_j(\bar{\lambda}) \in \mathscr{J}$ and the two blocks occur the same number of times.

We say that a collection of Jordan blocks $\mathscr{J}$ *corresponds* to the Jordan form of A provided the Jordan form of A is the direct sum of the elements in $\mathscr{J}$.

Let $\mathscr{J}$ be a collection of Jordan blocks and let J be the direct sum of the elements in $\mathscr{J}$. We define the *spectrum of* $\mathscr{J}$ by $\sigma(\mathscr{J}) = \sigma(J)$, the *spectral radius of* $\mathscr{J}$ by $\rho(\mathscr{J}) = \rho(J)$, the *peripheral spectrum of* $\mathscr{J}$ by $\pi(\mathscr{J}) = \pi(J)$, and the λ*-height characteristic of* $\mathscr{J}$ by $\eta_\lambda(\mathscr{J}) = \eta_\lambda(J)$.

A matrix $A \in \Re^{nn}$ is called:
positive ($A \gg 0$) if $a_{ij} > 0$, for all $i, j \in \langle n\rangle$;
semipositive ($A > 0$) if $a_{ij} \geqslant 0$, for all $i, j \in \langle n\rangle$ and $A \neq 0$; and
nonnegative ($A \geqslant 0$) if $a_{ij} \geqslant 0$, for all $i, j \in \langle n\rangle$.

Let $\Gamma = (V, E)$ be a (directed) graph, where V is a finite vertex set and $E \subseteq V \times V$ is an edge set. A *path* from j to l in Γ is a sequence of vertices $j = r_1, r_2, \ldots, r_t = l$, with $(r_i, r_{i+1}) \in E$, for $i = 1, \ldots, t-1$. A path for which the vertices are pairwise distinct is called a *simple path*. The empty path will be considered to be a simple path linking every vertex to itself.

We define the *graph* of A by $G(A) = (V, E)$, where $V = \langle n\rangle$ and $E = \{(i, j) | a_{ij} \neq 0\}$.

Let $\Gamma = (V, E)$ be a graph. If there is a path from a vertex j to a vertex l in Γ, we say that j has *access* to l. If j has access to l and l has access to j, we say j and l *communicate*. The communication relation is an equivalence relation, hence we may partition V into equivalence classes, which we will refer to as the (*irreducible*) *classes* of Γ.

Let $K, L \subseteq \langle n\rangle$. We will write A_{KL} to represent the submatrix of A whose rows are indexed from K and whose columns are indexed from L. If $\kappa = (K_1, K_2, \ldots, K_k)$ is an ordered partition of $\langle n\rangle$, we write

$$A_\kappa = \begin{bmatrix} A_{K_1K_1} & A_{K_1K_2} & \cdots & A_{K_1K_k} \\ A_{K_2K_1} & A_{K_2K_2} & \cdots & A_{K_2K_k} \\ \vdots & \vdots & & \vdots \\ A_{K_kK_1} & A_{K_kK_2} & \cdots & A_{K_kK_k} \end{bmatrix}.$$

We say A_κ is *block lower triangular* if $A_{K_iK_j} = 0$ whenever $i < j$. We refer to the blocks $A_{K_iK_j}$ as *subdiagonal* blocks whenever $i > j$. Given a matrix A, it is well known that there is an ordered partition $\kappa = (K_1, K_2, \ldots, K_k)$ of $\langle n\rangle$ so that each K_i corresponds to a class of $G(A)$

and A_κ is block lower triangular. We say that A_κ is the *Frobenius normal form* of A. A class K_j is said to be *singular* if $A_{K_jK_j}$ is singular, and *nonsingular* otherwise.

We define the *reduced graph of* A by $\mathscr{R}(A) = (V, E)$, where $V = \{K|K \text{ is a class of } A\}$, and $E = \{(K, L)| \text{ there is edge from a vertex } j \in K \text{ to a vertex } l \in L \text{ in } G(A)\}$.

The *singular length* of a simple path in $\mathscr{R}(A)$ is the sum of the indexes of zero of each of the singular vertices it contains. The *level* of a vertex K is the maximum singular length over all the simple paths in $\mathscr{R}(A)$ which terminate at K.

Let $\nu_i(A)$ be the number of singular vertices with level i in $\mathscr{R}(A)$ and let m be the largest number for which $\nu_i(A) \neq 0$. Then $\nu(A) = (\nu_1(A), \dots, \nu_m(A))$ is referred to as the *level characteristic* of A.

Let $\kappa = (K_1, \dots, K_k)$ correspond to the Frobenius normal form of a nonnegative matrix A. Let $\rho = \rho(A)$ and $m = \text{index}_\rho(A)$. Set

$$M_j = \cup\{K_l|\text{the level of } K_l \text{ is } m+1-j \text{ in } \mathscr{R}(\rho I - A)\}, \quad j \in \langle m+1\rangle.$$

Then $\mu = (M_1, M_2, \dots, M_m, M_{m+1})$ is referred to as the *level partition of A with respect to the eigenvalue* $\rho(A)$, M_q is referred to as a *level set*, and A_μ is referred to as a *level form of* A. We note that our subscripting matches that of [3], but is different from [6]. Our definition also differs from that of [6] in that it includes the nonsingular classes.

In addition, we see that for any $j \in \langle m\rangle$, M_j can be further partitioned into two (not necessarily nonempty) subsets. We set

$$L_{2j} = \cup\{K_l| \text{ the level of } K_l \text{ is } m+1-j \text{ in } \mathscr{R}(\rho I - A) \text{ and } \rho(A_{K_lK_l}) = \rho\}$$

and

$$L_{2j-1} = M_j \backslash L_{2j}.$$

Notice then that

(i) $\rho(A_{L_{2j-1}L_{2j-1}}) < \rho$.
(ii) $A_{L_{2j}L_{2j}}$ is the direct sum of blocks whose spectral radius is ρ.
(iii) $A_{L_{2j-1}L_{2j}} = 0$.

We set $L_{2m+1} = M_{m+1}$ and refer to $\Lambda = (L_1, L_2, \dots, L_{2m+1})$ as the *split–level partition of* A *with respect to the eigenvalue* $\rho(A)$. We will refer to L_q as a *split-level set*. Notice that A_μ and A_Λ are block lower triangular. We say that A_Λ is in *split-level form*.

Definition 2.1. Let $\mathscr{J}$ be a self-conjugate collection of Jordan blocks all of whose eigenvalues have modulus 1. Let m be the size of the largest Jordan block in $\mathscr{J}$. We say $\mathscr{J}$ satisfies the *extended Tam–Schneider condition* provided that there is a sequence of multisets $\mathscr{J}_1, \dots, \mathscr{J}_m$ such that

(i) $J_m(1) \in \mathscr{J}$ and $\mathscr{J}_m = \mathscr{J}$.
(ii) $\mathscr{J}_1$ is a collection of 1×1 Jordan blocks which can be partitioned into complete sets of roots of unity.
(iii) For any $2 \leqslant j \leqslant m$, if we enumerate the blocks in $\mathscr{J}_j$ as $J^{(1)}, J^{(2)}, \dots, J^{(r)}$ and create the sets

$$S_j = \{(t, \lambda)|\lambda \text{ is an eigenvalue of } J^{(t)} \in \mathscr{J}_j, \text{ where } J^{(t)} \text{ is } j \times j\},$$

and

$$T_j = \{(t, \lambda)|\lambda \text{ is an eigenvalue of } J^{(t)} \in \mathscr{J}_j, \text{ where } J^{(t)} \text{ is } p \times p \text{ with } p < j\},$$

then there is a set $U_j \subseteq T_j$ so that

(a) $\mathscr{J}_{j-1}$ can be formed from $\mathscr{J}_j$ by removing the first row and column from each Jordan block in $\mathscr{J}_j$ labelled with an element appearing as a first coordinate in $S_j \cup U_j$ and leaving all other Jordan blocks the same. Note that if we remove the first row and column of a 1×1 block, then we simply remove the block itself.

(b) The second coordinates of $S_j \cup U_j$ can be partitioned into complete sets of roots of unity.

3. The peripheral spectrum of a nonnegative matrix

It is well known that the peripheral spectrum of a nonnegative matrix is a union of complete sets of roots of unity multiplied the spectral radius of the matrix. We expect the following result to be known, but offer a proof for completeness.

Lemma 3.1. *If a multiset S can be partitioned into complete sets of roots of unity, then the partition is unique.*

Proof. Note that the number of times 1 is listed in S determines the number of sets in the partition of S into complete sets of roots of unity. We induct on this number. If S partitions into one complete set of roots of unity, the result holds. Suppose S partitions into t sets of roots of unity. Suppose each element of S is written in the form $e^{\frac{2\pi ik}{n}}$ where $\frac{k}{n} < 1$ and $\gcd(k, n) = 1$. Let $\alpha = \max\left\{\frac{k}{n} : e^{\frac{2\pi ik}{n}} \in S\right\}$. Note that $\alpha = \frac{m-1}{m}$ for some $m \in \mathbf{N}$. Then $\mathbf{Z}_m$ must be a set in the partition of S into complete sets of roots of unity. Otherwise there is an integer $p > 1$ such that $\mathbf{Z}_{pm}$ is a set in the partition so $e^{\frac{2\pi i(pm-1)}{pm}} \in S$, but $\frac{pm-1}{pm} > \frac{m-1}{m}$, a contradiction. Then $S \setminus \mathbf{Z}_m$ partitions into complete sets of roots of unity and by the inductive hypothesis, this partition is unique. Hence, the partition of S into complete sets of roots of unity is unique. □

Notice that the above proof suggests an algorithm for determining whether or not a set S partitions into complete sets of roots of unity, and if so, determining the sets in the partition. This is discussed further in the following section where an algorithm for determining whether or not a multiset of Jordan blocks corresponds to the peripheral spectrum of a nonnegative matrix is outlined. The relationship between the height characteristic and the level characteristic of a nonnegative matrix with respect to the Perron eigenvalue has played an important role in the study of reducible nonnegative matrices. Here, we illustrate how these ideas can be generalized to the entire peripheral spectrum.

Definition 3.2. Suppose $A \geqslant 0$. Let $\mu = (M_1, M_2, \ldots, M_m, M_{m+1})$ be the level partition of A with respect to $\rho = \rho(A)$. For each $\lambda \in \pi(A)$, let $\nu_i(\lambda)$ be the number of times λ occurs as an eigenvalue of $A_{M_iM_i}$. Then the $\lambda-$ *level characteristic of* A (*with respect to* ρ) is the sequence $\nu_{\lambda,\rho}(A) = (\nu_1(\lambda), \ldots, \nu_m(\lambda))$ (note that $\nu_{m+1}(\lambda) = 0$ so it need not be included).

The corollary below follows easily from Lemma 3.2 of [5], which we restate here.

Lemma 3.3 [5, Lemma 3.2]. *Let A be a nonnegative matrix. Set* $m = \text{index}_{\rho(A)}(A)$ *and let* $(L_1, L_2, \ldots, L_{2m+1})$ *be the split-level partition of A with respect to* $\rho(A)$. *Set*

$$P_j = \bigcup_{q=2(m+1-j)}^{2m+1} L_q$$

and let $\lambda \in \pi(A)$. *Then for* $j = 2, \ldots, m$, *the Jordan form of* λ *for* $A_{P_j P_j}$ *can be produced from the Jordan form of* λ *for* $A_{P_{j-1} P_{j-1}}$ *by increasing the size of a select number of Jordan blocks by one, and adding copies of* $J_1(\lambda)$.

Corollary 3.4. *Let* $A \geqslant 0$ *with* $\rho(A) = \rho$. *Then, for each* $\lambda \in \pi(A)$, $\hat{\nu}_{\lambda,\rho}(A) \preceq \eta_\lambda(A)$.

As stated in the above definition and corollary, the peripheral eigenvalues of a nonnegative matrix A must be distributed among the levels of A such that the majorization condition in Corollary 3.4 is satisfied. We are interested in the question of whether or not there exists a nonnegative matrix with peripheral spectrum corresponding to a given multiset $\mathscr{J}$ of Jordan blocks. Corollary 3.4 asserts that the eigenvalues of $\mathscr{J}$ must partition into level sets satisfying the majorization criterion. The definition below will allow us to gather information about a partition of a multiset of eigenvalues.

Definition 3.5. Suppose $L_1, \ldots, L_k$ are multisets of eigenvalues. For each $\lambda \in \bigcup_{i=1}^k L_i$, let $\nu_\lambda(L_i)$ be the number of times λ is listed in L_i. Then $\nu_\lambda(L_1, L_2, \ldots, L_k)$ is defined to be the sequence $(\nu_\lambda(L_1), \nu_\lambda(L_2), \ldots, \nu_\lambda(L_k))$.

The following lemmas will be used in the proof of the main theorem.

Lemma 3.6. *Let* α *and* β *be decreasing sequences of length* k *satisfying* $\alpha \preceq \beta$. *Then* $\alpha_j \geqslant \beta_k$ *for* $j = 1, \ldots, k$.

Proof. $\Sigma_{i=1}^k \alpha_i = \Sigma_{i=1}^k \beta_i$ and $\Sigma_{i=1}^{k-1} \alpha_i \leqslant \Sigma_{i=1}^{k-1} \beta_i$ implies $\alpha_j \geqslant \alpha_k \geqslant \beta_k$. $\square$

Lemma 3.7. *Suppose* α *and* β *are decreasing sequences of length* k *satisfying* $\alpha \preceq \beta$. *Let*

$$\tilde{\alpha}(j) = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_k)$$

and

$$\tilde{\beta}(j) = (\tilde{\beta}_1, \ldots, \tilde{\beta}_{k-1}),$$

where

$$\tilde{\beta}_i = \beta_i - \max\{0, \alpha_j - \Sigma_{t=i+1}^k \beta_t\}.$$

Then $\tilde{\alpha}(j) \preceq \tilde{\beta}(j)$.

Proof. If $s < j$, then $\alpha_s = [\tilde{\alpha}(j)]_s$ and $\beta_s = \tilde{\beta}_s$. If $s \geqslant j$, then $\sum_{i=1}^s [\tilde{\alpha}(j)]_i = -\alpha_j + \sum_{i=1}^{s+1} \alpha_i \leqslant -\alpha_j + \sum_{i=1}^{s+1} \beta_i \leqslant \sum_{i=1}^s \tilde{\beta}_i$. Also, $\Sigma_{i=1}^{k-1} [\tilde{\alpha}(j)]_i = -\alpha_j + \Sigma_{i=1}^k \alpha_i = -\alpha_j + \Sigma_{i=1}^k \beta_i = \Sigma_{i=1}^{k-1} \tilde{\beta}_i$. $\square$

We are now ready to state and prove our main theorem. Note that we state our theorem for the case where $\rho(A) = 1$. If $\rho(A) \neq 1$, consider the matrix $\frac{1}{\rho(A)} A$.

Theorem 3.8. *Let* $\mathscr{J}$ *be a self conjugate multiset of Jordan blocks all of whose eigenvalues have modulus* 1. *Let* m *be the size of the largest Jordan block in* $\mathscr{J}$ *with eigenvalue* 1. *Then the following are equivalent*

(i) $\mathscr{J}$ *corresponds to the peripheral Jordan form of a nonnegative matrix.*
(ii) $\sigma(\mathscr{J}) = L_1 \bigcup L_2 \bigcup \cdots \bigcup L_m$ *where* $\bigcup$ *represents a multiset union and*

(a) *each L_i partitions into complete sets of roots of unity and*
(b) *for each $\lambda \in \sigma(\mathscr{J})$, $\hat{\nu}_\lambda(L_1, \ldots, L_m) \preceq \eta_\lambda(\mathscr{J})$.*

Proof. (i) $\Rightarrow$ (ii): Suppose $\mathscr{J}$ corresponds to the peripheral Jordan form of a nonnegative matrix A. Let $(M_1, \ldots, M_{m+1})$ be the level partition of A with respect to 1. Let $L_i = \pi(A_{M_i M_i})$ for $i = 1, \ldots, m$. Note that $\bigcup_{i=1}^m L_i = \pi(A) = \sigma(\mathscr{J})$. By [5, Lemma 3.1] (iii), each L_i is a union of complete sets of roots of unity. By Corollary 3.4, $\hat{\nu}_{\lambda,\rho}(A) = \hat{\nu}_\lambda(L_1, \ldots, L_m) \preceq \eta_\lambda(A) = \eta_\lambda(\mathscr{J})$.

(ii) $\Rightarrow$ (i): Set $\mathscr{J}_m = \mathscr{J}$ and let sets S_m and T_m be defined as in Definition 2.1. For each $\lambda \in \sigma(\mathscr{J})$, set $\eta_\lambda = \eta_\lambda(\mathscr{J}_m)$. Notice that $(\hat{\nu}_\lambda)_j \leqslant (\hat{\nu}_\lambda)_1 \leqslant (\eta_\lambda)_1$ for all $j \in \langle m \rangle$, so the number of λ in L_m is less than or equal to the number of Jordan blocks in $\mathscr{J}$ with eigenvalue λ. Using Lemma 3.6, $(\hat{\nu}_\lambda)_j \geqslant (\eta_\lambda)_m$, so the number of λ in L_m is at least the number of Jordan blocks of size m in $\mathscr{J}$. Hence, $U_m \subset T_m$ may be chosen so that the second coordinates of the elements of $S_m \bigcup U_m$ are the elements of L_m. If there is a choice, we choose the largest Jordan block(s) to be represented in U_m. Create $\mathscr{J}_{m-1}$ by removing one row and one column from each Jordan block represented in $S_m \bigcup U_m$ and leaving all others unchanged.

By Lemma 3.7, for each $\lambda \in \sigma(\mathscr{J})$, $\hat{\nu}_\lambda(L_1, \ldots, L_{m-1}) \preceq \eta_\lambda(\mathscr{J}_{m-1})$ so $U_{m-1} \subset T_{m-1}$ may be chosen so that the second coordinates of the elements of $S_{m-1} \bigcup U_{m-1}$ are the elements of L_{m-1}. We continue in this manner, noting that for each $\lambda \in \sigma(\mathscr{J})$ and $j = 1 \ldots (m-1)$, $\hat{\nu}_\lambda(L_1, \ldots, L_{m-j}) \preceq \eta_\lambda(\mathscr{J}_{m-j})$ provided that we remove rows and columns from the largest remaining Jordan block(s) when there is a choice.

Then $\mathscr{J}$ satisfies the Extended Tam–Schneider condition so by [5, Theorem 3.5], $\mathscr{J}$ corresponds to the peripheral Jordan form of a nonnegative matrix. $\square$

Observation 3.9. Suppose $\mathscr{J}$ satisfies the Extended Tam-Schneider condition. Then, as shown in the proof of [5, Theorem 3.5], a nonnegative matrix A in level form can be constructed such that $\pi(A_{M_q M_q})$ consists of the second coordinates of $S_i \bigcup U_i$ for $i \in \langle m \rangle$ where $\mu = (M_1, \ldots, M_{m+1})$ is the level partition of A with respect to $\rho(A)$ and $q = m + 1 - i$. From the proof of Theorem 3.8, L_i can be chosen to consist of the second coordinates of $S_i \bigcup U_i$. Hence, $\pi(A_{M_q M_q}) = L_i$ where $q = m + 1 - i$ for $i \in \langle m \rangle$.

The above observation leads to the following corollary.

Corollary 3.10. *Let*

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}$$

be a nonnegative matrix in level form, all of whose eigenvalues have modulus 1. *Then, for any permutation τ of $\langle m \rangle$, there exists a $B \geqslant 0$ similar to A such that the level form of B is*

$$B = \begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ B_{21} & B_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix}.$$

and $\sigma(B_{ii}) = \sigma(A_{\tau(i)\tau(i)})$ for $i = 1, \ldots, m$.

Proof. Let $\mathscr{J}$ be the collection of Jordan blocks corresponding to the Jordan form of A. Let $L_i = \sigma(A_{qq})$ where $q = m + 1 - i$. Then by [5, Lemmas 3.1 and 3.2], we have m sets $L_1, L_2, \ldots, L_m$ satisfying the conditions in Theorem 3.8 (ii). But the sets $L_{\tau(1)}, L_{\tau(2)}, \ldots, L_{\tau(m)}$ also satisfy the conditions in Theorem 3.8 (ii), so we may construct a nonnegative matrix B with Jordan form corresponding to $\mathscr{J}$ and $\sigma(B_{qq}) = \pi(B_{qq}) = L_{\tau(i)} = \sigma(A_{\tau(q)\tau(q)})$ where $q = m + 1 - i$, $i = 1, \ldots, m$. Since A and B both have Jordan form corresponding to $\mathscr{J}$, A and B are similar. $\square$

We will refer to the condition stated in Theorem 3.8(ii) (b) as the majorization condition. Suppose $\mathscr{J}$ is a multiset of Jordan blocks and m is the size of the largest Jordan block in $\mathscr{J}$. Note that it is possible to find a partition $L_1, \ldots, L_m$ of $\sigma(\mathscr{J})$ such that the majorization condition is satisfied. As the next example illustrates, it may also be possible to partition the eigenvalues of $\mathscr{J}$ into m sets, each of which is a union of complete sets of roots of unity. However, it may not be possible to achieve both requirements with the *same* partition of $\sigma(\mathscr{J})$, and hence $\mathscr{J}$ does not correspond to the peripheral Jordan form of a nonnegative matrix.

Example 3.11. Consider the following multiset of Jordan blocks whose spectrum partitions into $\mathbf{Z}_6 \cup \mathbf{Z}_{10} \cup \mathbf{Z}_{15}$

$$\begin{aligned}\mathscr{J} = \Big\{ & J_2(1), J_2\left(e^{\frac{2\pi i}{2}}\right), J_2\left(e^{\frac{2\pi i}{3}}\right), J_2\left(e^{\frac{4\pi i}{3}}\right), J_2\left(e^{\frac{2\pi i}{5}}\right), J_2\left(e^{\frac{8\pi i}{5}}\right), J_1(1), J_1\left(e^{\frac{2\pi i}{6}}\right), J_1\left(e^{\frac{10\pi i}{6}}\right),\\ & J_1\left(e^{\frac{2\pi i}{10}}\right), J_1\left(e^{\frac{6\pi i}{10}}\right), J_1\left(e^{\frac{8\pi i}{10}}\right), J_1\left(e^{\frac{12\pi i}{10}}\right), J_1\left(e^{\frac{14\pi i}{10}}\right), J_1\left(e^{\frac{18\pi i}{10}}\right), J_1\left(e^{\frac{2\pi i}{15}}\right), J_1\left(e^{\frac{4\pi i}{15}}\right),\\ & J_1\left(e^{\frac{8\pi i}{15}}\right), J_1\left(e^{\frac{12\pi i}{15}}\right), J_1\left(e^{\frac{14\pi i}{15}}\right), J_1\left(e^{\frac{16\pi i}{15}}\right), J_1\left(e^{\frac{18\pi i}{15}}\right), J_1\left(e^{\frac{22\pi i}{15}}\right), J_1\left(e^{\frac{26\pi i}{15}}\right), J_1\left(e^{\frac{28\pi i}{15}}\right)\Big\}.\end{aligned}$$

It was shown in [5] Example 3.7 that $\mathscr{J}$ does not correspond to the peripheral spectrum of a nonnegative matrix. We see that the eigenvalues partition into complete sets of roots of unity, but these sets of roots cannot be partitioned into $m = 2$ levels so that the majorization condition is satisfied. Note that the partitions of $\sigma(\mathscr{J})$ into 2 sets, each of which is a union of complete sets of roots of unity are as follows:

(a) $L_1 = \mathbf{Z}_{15}$, $L_2 = \mathbf{Z}_{10} \bigcup \mathbf{Z}_6$ or
(b) $L_1 = \mathbf{Z}_{10}$, $L_2 = \mathbf{Z}_{15} \bigcup \mathbf{Z}_6$ or
(c) $L_1 = \mathbf{Z}_6$, $L_2 = \mathbf{Z}_{15} \bigcup \mathbf{Z}_{10}$.

Then $\eta_\lambda(\mathscr{J}) = (1, 1)$ does not majorize $\hat{\nu}_\lambda(L_1, L_2) = (2, 0)$ for $\lambda = e^{\frac{2\pi i}{2}}$, $e^{\frac{2\pi i}{3}}$, and $e^{\frac{2\pi i}{5}}$, respectively.

Note that it is possible to partition $\sigma(\mathscr{J})$ into 2 sets, L_1 and L_2 such that, for each $\lambda \in \sigma(\mathscr{J})$, $\hat{\nu}_\lambda(L_1, L_2) \preceq \eta_\lambda$. For example, consider $L_1 = \mathbf{Z}_{15} \bigcup \left\{e^{\frac{2\pi i}{2}}\right\}$ and $L_2 = \mathbf{Z}_{10} \bigcup \mathbf{Z}_6 \setminus \left\{e^{\frac{2\pi i}{2}}\right\}$. However, each L_i is not a union of complete sets of roots of unity.

4. An algorithmic approach

In this section, we present a recursive branch and bound algorithm based on Theorem 3.8 that will determine whether or not a multiset of eigenvalues $\mathscr{J}$ corresponds to the peripheral Jordan form of a nonnegative matrix. If $\mathscr{J}$ does correspond to the peripheral Jordan form of a nonnegative

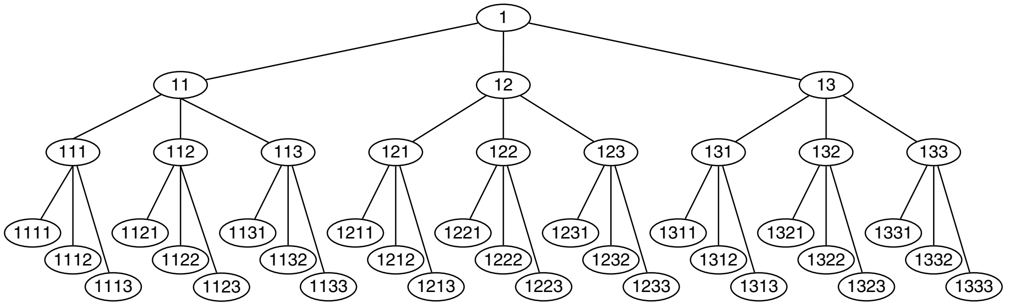

Fig. 1. Search tree for $m = 3$ and $\sigma(\mathscr{J}) = \mathbf{Z}_{R_1} \bigcup \mathbf{Z}_{R_2} \bigcup \mathbf{Z}_{R_3} \bigcup \mathbf{Z}_{R_4}$.

matrix, two vectors, R and N are returned indicating that a partition of $\sigma(\mathscr{J})$ into sets $L_1, \ldots, L_m$ can be formed by placing $\mathbf{Z}_{R_i}$ in L_{N_i}. We will refer to N as the assignment vector (as each entry of N assigns a set of roots to one of the sets $L_1, \ldots, L_m$). Pseudocode for the main function is as follows:

LEVEL($\mathscr{J}$)
$R \leftarrow$ PARTITION($\mathscr{J}$)
$N \leftarrow [1]$,
$N \leftarrow$ PLACE_ROOTS($R, N, \mathscr{J}$)
return R, N

The function PARTITION determines whether or not $\sigma(\mathscr{J})$ can be partitioned into complete sets of roots of unity. If so, the vector R is returned. The function works recursively as indicated in the proof of Lemma 3.1:

PARTITION($\mathscr{J}$)
while $\sigma(\mathscr{J}) \neq \emptyset$

$$n \leftarrow \max\left\{n : \mathrm{e}^{\frac{2\pi \mathrm{i}k}{n}} \in \sigma(\mathscr{J})\right\}$$

if $\mathbf{Z}_n \subset \sigma(\mathscr{J})$

$\sigma(\mathscr{J}) \leftarrow \sigma(\mathscr{J}) \setminus \mathbf{Z}_n$
$R \leftarrow [R, n]$

else error (eigenvalues do not partition into complete sets of roots of unity)

return R

The function PLACE_ROOTS is a recursive function that determines whether or not the sets of roots of unity specified in R can be placed in m sets satisfying the majorization condition. If so, the vector N is returned. We begin with $N = [N_1] = [1]$ indicating that we will place $\mathbf{Z}_{R_1}$ in L_1. We then place $\mathbf{Z}_{R_2}$ in either L_1 (if the majorization condition can still be satisfied) or L_2 (otherwise). We continue in this manner. After Z_{R_j} has been placed in some L_i, we place $Z_{R_{j+1}}$ in L_k where k is the smallest integer in $\langle m\rangle$ such that $\sum_{i=1}^{j}[\hat{\nu}_\lambda(L_1, \ldots, L_m)]_i \leqslant \sum_{i=1}^{j}[\eta_\lambda]_i$ for $j \in \langle m\rangle$. If no such k exists, then we prune the current branch of the search tree and proceed to the next vertex. Note that $\sum_{i=1}^{m}[\hat{\nu}_\lambda(L_1, \ldots, L_m)]_i = \sum_{i=1}^{m}[\eta_\lambda]_i$ once all sets of roots of unity have been placed. A sample search tree for the case when $\sigma(\mathscr{J}) = \mathbf{Z}_{R_1} \bigcup \mathbf{Z}_{R_2} \bigcup \mathbf{Z}_{R_3} \bigcup \mathbf{Z}_{R_4}$ and $m = 3$ is given in Fig. 1.

Since the numbering of the sets $L_1, \ldots, L_m$ is arbitrary, placement of $\mathbf{Z}_{R_1}$ in any other L_i need not be considered. Moreover, we only consider N satisfying $N_i \leqslant 1 + \max_{j<i} N_j$ for $i \in \langle m\rangle$

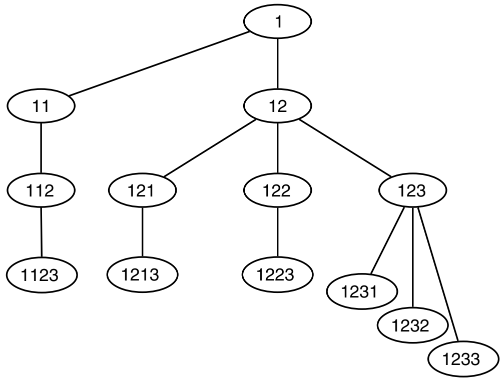

Fig. 2. Pruned search tree for $m = 3$ and $\sigma(\mathscr{J}) = \mathbf{Z}_{R_1} \bigcup \mathbf{Z}_{R_2} \bigcup \mathbf{Z}_{R_3} \bigcup \mathbf{Z}_{R_4}$.

since we do not assign a set of roots to L_i if none have yet been assigned to L_{i-1}. Also, at least one set of roots must be assigned to each L_i, otherwise the majorization condition will be violated for $\lambda = 1$. So if $\max(N) < m - \text{length}(R) + \text{length}(N)$, then N is not a valid assignment vector so we move to the next assignment vector. This allows us to prune many branches of the search tree quickly. For example, the only vertices remaining after pruning the tree in Fig. 1 are shown in Fig. 2.

PLACE_ROOTS(R, N, $\mathscr{J}$)
if $N_1 = 2$

  error (no level assignments satisfy the majorization condition)

$k \leftarrow \text{length}(N)$
if $N_k > 1 + \max_{j<k} N_j$ or $\max_{j\leqslant k} N_j < m - \text{length}(R) + \text{length}(N)$

  $N \leftarrow$ BYPASS_CHILDREN(N)

  PLACE_ROOTS(R, N, $\mathscr{J}$)

if majorization condition can be/is satisfied

  if length(N) = length(R)

    return R, N

  else $N \leftarrow$ NEXT_VERTEX(N)

else $N \leftarrow$ BYPASS_CHILDREN(N)
PLACE_ROOTS (R, N, $\mathscr{J}$)

NEXT_VERTEX is a function which proceeds to the next vertex in the pre-ordered search tree. BYPASS_CHILDREN is used when the current N is not a valid assignment vector (so no assignment vector beginning with the entries of N will be valid). In this case, we move to the next vertex in the search tree which does not begin with the current assignment vector N. Once all of the vertices in the search tree have been considered or pruned, both NEXT_VERTEX and BYPASS_CHILDREN return the assignment vector $N = [2]$ which indicates that $\mathscr{J}$ does not correspond to the peripheral Jordan form of a nonnegative matrix. Both functions are modeled after the pseudocode given in [4, pp. 107–108].

The MATLAB code for this algorithm is available on the authors' websites.

Acknowledgments

We would like to thank the referee for his/her comments and suggestions.

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